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Generalized Invexity and Duality in Multiobjective Programming Problems

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Abstract. In this paper, we are concerned with the multiobjective programming problem with inequality constraints. We introduce new classes of generalized type I vector-valued functions. Duality theorems are proved for Mond–Weir and general Mond–Weir type duality under the above generalized type I assumptions.

Key words: Multiobjective programming, Efficient solution, Convexity, Invexity, Type I, Weak, Strong, Converse duality

1. Introduction

In optimization theory, convexity plays an important role in deriving sufficient conditions and duality for the nonlinear programming problem. Various generalizations of convexity have been made in the literature. Hanson (1981) introduced the class of invex functions. Later Hanson and Mond (1987) defined two new classes of functions called type I and type II functions, and sufficient optimality conditions were established involving these generalized functions. In Kaul, Suneja and Srivastava (1994) consider a multiobjective nonlinear programming problem involving type I functions to obtain some duality results, where Wolfe and Mond–Weir duals are considered. Recently, Giorgi and Guerraggio (1998) have generalized the notion of invexity to vector-valued functions and they provided some duality results. In this paper, we introduce new classes of vector-valued functions and derive various duality results for the nonlinear multiobjective programming problem. To establish our results with our classes of functions we do not require the assumption of the scalarization of the objective functions as is done in (Giorgi and Guerraggio (1998), Kaul et al. (1994)). Consider the following multiobjective optimization problem:

$$
\begin{aligned}
(MOP) \qquad & \text{minimize} \quad f(x) = (f_1(x), \ldots, f_p(x)) \\
& \text{subject to} \quad g(x) \leqq 0, \\
& \qquad\qquad\quad x \in X (\subseteq \mathbb{R}^n), \qquad X \ \text{open},
\end{aligned}
$$

where $f : X \longrightarrow \mathbb{R}^p$ and $g : X \longrightarrow \mathbb{R}^m$ are differentiable functions on a set $X \subseteq \mathbb{R}^n$ and minimization means obtaining efficient solutions for the problem (MOP).

For any $x = (x_1, x_2, ..., x_n)^t$, $y = (y_1, y_2, ..., y_n)^t \in \mathbb{R}^n$, We denote:

$$\begin{array}{llll} x = y & \text{implying} & x_i = y_i, & i = 1, ..., n; \\ x \leqq y & \text{implying} & x_i \leq y_i, & i = 1, ..., n; \\ x \leq y & \text{implying} & x \leqq y, \quad \text{and} & x \neq y; \\ x < y & \text{implying} & x_i < y_i, & i = 1, ..., n. \end{array}$$

Let

$$\begin{array}{ll} A = \{x \in X,\ g(x) \leqq 0\}, & I = \{j :\ g_j(\overset{\circ}{x}) = 0\} \\ M = \{1, 2, \ldots, m\}, & P = \{1, 2, \ldots, p\}. \end{array}$$

For such multicriterion optimization problems, the solution is defined in terms of a (weak) efficient (Pareto minimal) solution in the following sense

DEFINITION 1.1. *We say that $\overset{\circ}{x} \in A$ is an efficient solution for (MOP) if and only if there exists no $x \in A$ such that $f(x) \leq f(\overset{\circ}{x})$.*

DEFINITION 1.2. *We say that $\overset{\circ}{x} \in A$ is a weak efficient solution for (MOP) if and only if there exists no $x \in A$ such that $f(x) < f(\overset{\circ}{x})$.*

In the first half of this paper, we consider a nonlinear multiobjective programming problem with inequality constraints and we introduce new classes of generalized type I vector-valued functions. In the second half, Mond–Weir and generalized Mond–Weir type duals are formulated and the concept of efficiency is used to state some duality results under generalized type I assumptions.

2. Preliminaries

It will be assumed throughout that f is the vector objective function and g is the constraint vector function in problem (MOP). The definition of type I for single objective and constraint vector (Hanson and Mond (1987)) can be generalized easily to a multiple objective and constraint vector.

DEFINITION 2.1. *(f, g) is said to be type I with respect to η at $\overset{\circ}{x} \in X$ if there exists a vector function $\eta(x, \overset{\circ}{x})$ defined on $A \times X$ such that, for all $x \in A$,*

$$f(x) - f(\overset{\circ}{x}) \geqq (\nabla f(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}), \tag{2.1}$$

$$-g(\overset{\circ}{x}) \geqq (\nabla g(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}). \tag{2.2}$$

If in the above definition, (2.1) is a strict inequality, then we say that (f, g) is semistrictly-type I at $\overset{\circ}{x}$.

We now define and introduce the notions of weak strictly-pseudoquasi-type I, strong pseudoquasi-type I, weak quasistrictly-pseudo-type I and weak strictly pseudo-type I functions for (MOP).

DEFINITION 2.2. (f, g) *is said to be weak strictly-pseudoquasi-type I with respect to* η *at* $\overset{\circ}{x} \in X$ *if there exists a vector function* $\eta(x, \overset{\circ}{x})$ *defined on* $A \times X$ *such that, for all* $x \in A$,

$$f(x) \leq f(\overset{\circ}{x}) \;\Rightarrow\; (\nabla f(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) < 0, \tag{2.3}$$

$$-g(\overset{\circ}{x}) \leqq 0 \;\Rightarrow\; (\nabla g(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) \leqq 0. \tag{2.4}$$

This definition is a slight extension of that of the strictly pseudoquasi-type I functions (Kaul et al. (1994)). This class of functions does not contain the class of type I functions, but does contain the class of semistrictly-type I functions.

DEFINITION 2.3. (f, g) *is said to be strong pseudoquasi-type I with respect to* η *at* $\overset{\circ}{x} \in X$ *if there exists a vector function* $\eta(x, \overset{\circ}{x})$ *defined on* $A \times X$ *such that, for all* $x \in A$,

$$f(x) \leq f(\overset{\circ}{x}) \;\Rightarrow\; (\nabla f(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) \leq 0, \tag{2.5}$$

$$-g(\overset{\circ}{x}) \leqq 0 \;\Rightarrow\; (\nabla g(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) \leqq 0. \tag{2.6}$$

Instead of the class of weak strictly-pseudoquasi-type I, the class of strong pseudoquasi-type I functions does contain the class of type I.

We give examples to show that weak strictly-pseudoquasi-type I and strong pseudoquasi-type I functions exist. Weak strictly-pseudoquasi-type I functions need not be strictly-pseudoquasi-type I for the same $\eta(x, \overset{\circ}{x})$ as can be seen from the following example.

EXAMPLE 2.1. *The functions* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $f(x) = (x_1 \exp(\sin x_2), x_2(x_2 - 1)\exp(\cos x_1))$ *and* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $g(x) = 2x_1 + x_2 - 2$ *are weak strictly pseudoquasi-type I with respect to* $\eta(x, \overset{\circ}{x}) = (x_1 + x_2 - 1, x_2 - x_1)$ *at* $\overset{\circ}{x} = (0, 0)$ *but* $f(x)$ *and* $g(x)$ *are not strictly-pseudoquasi-type I with respect to the same* $\eta(x, \overset{\circ}{x})$ *at* $\overset{\circ}{x}$ *because for* $x = (0, 1)$ *and* $\overset{\circ}{x} = (0, 0)$

$$f(x) \leqq f(\overset{\circ}{x}) \qquad \textit{but} \qquad \nabla f(\overset{\circ}{x})\eta(x, \overset{\circ}{x}) \not< (0, 0),$$

also f *and* g *are not type I with respect to the same* $\eta(x, \overset{\circ}{x})$ *at* $\overset{\circ}{x}$ *as can be seen by taking* $x = (-\pi/2, 1/2)$.

Strong pseudoquasi-type I functions need not be Type I with respect to the same $\eta(x, \overset{\circ}{x})$.

EXAMPLE 2.2. *The functions* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $f(x) = (x_1(x_1 - 1)^2, x_2(x_2-1)^2(x_2^2+2))$ *and* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $g(x) = x_1^2 + x_2^2 - 9$ *are strong pseudoquasi-type I with respect to* $\eta(x, \overset{\circ}{x}) = (x_1 - 1, x_2 - 1)$ *at* $\overset{\circ}{x} = (0, 0)$ *but* $f(x)$ *and* $g(x)$ *are not type I with respect to the same* $\eta(x, \overset{\circ}{x})$ *as can be seen by taking* $x = (0, -2)$ *nor they are weak strictly-pseudoquasi-type I with respect to the same* $\eta(x, \overset{\circ}{x})$ *as can be seen by taking* $x = (1, -1)$.

DEFINITION 2.4. (f, g) *is said to be weak quasistrictly-pseudo-type I with respect to* η *at* $\overset{\circ}{x} \in X$ *if there exists a vector function* $\eta(x, \overset{\circ}{x})$ *defined on* $A \times X$ *such that, for all* $x \in A$,

$$f(x) \leq f(\overset{\circ}{x}) \;\Rightarrow\; (\nabla f(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) \leqq 0, \tag{2.7}$$

$$-g(\overset{\circ}{x}) \leqq 0 \;\Rightarrow\; (\nabla g(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) \leq 0. \tag{2.8}$$

EXAMPLE 2.3. *The functions* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $f(x) = (x_1^3(x_1^2 + 1), x^2(x_2-1)^3)$ *and* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $g(x) = ((2x_1 - 4)\exp(-x_2^2), (x_1 + x_2 - 2)(x_1^2 + 2x_1 + 4))$ *are weak quasistrictly-pseudo-type I with respect to* $\eta(x, \overset{\circ}{x}) = (x_1, x_2(1 - x_2))$ *at* $\overset{\circ}{x} = (0, 0)$ *but* $f(x)$ *and* $g(x)$ *are not type I with respect to the same* $\eta(x, \overset{\circ}{x})$ *as can be seen by taking* $x = (-1, 0)$.

DEFINITION 2.5. (f, g) *is said to be weak strictly pseudo-type I with respect to* η *at* $\overset{\circ}{x} \in X$ *if there exists a vector function* $\eta(x, \overset{\circ}{x})$ *defined on* $A \times X$ *such that, for all* $x \in A$,

$$f(x) \leq f(\overset{\circ}{x}) \;\Rightarrow\; (\nabla f(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) < 0, \tag{2.9}$$

$$-g(\overset{\circ}{x}) \leqq 0 \;\Rightarrow\; (\nabla g(\overset{\circ}{x}))\eta(x, \overset{\circ}{x}) < 0. \tag{2.10}$$

EXAMPLE 2.4. *The functions* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $f(x) = (x_1(x_1^2 + 1), x_2(x_2 - 1)(x_2^2 + 2))$ *and* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *defined by* $g(x) = x_1(x_2^2 + 1)$ *are strong pseudoquasi-type I with respect to* $\eta(x, \overset{\circ}{x}) = (x_1 + x_2 - 1, x_2 - 1)$ *at* $\overset{\circ}{x} = (0, 0)$ *but* $f(x)$ *and* $g(x)$ *are not type I with respect to the same* $\eta(x, \overset{\circ}{x})$ *as can be seen by taking* $x = (-2, 0)$.

3. Mond–Weir Vector Duality

In this section we give some weak, strong, and converse duality relations between problems (MOP) and (DMOP). We consider the following Mond-Weir dual

(DMOP) suggested by Egudo (1989) for problem (MOP).

$$(DMOP) \quad \text{maximize} \quad f(y),$$

$$\text{subject to} \quad u\nabla f(y) + v\nabla g(y) = 0, \tag{3.11}$$

$$vg(y) \geqq 0, \tag{3.12}$$

$$v \geqq 0, \tag{3.13}$$

$$u \geqq 0, \tag{3.14}$$

$$ue = 1; \tag{3.15}$$

where $e = (1, 1, ..., 1)^t \in \mathbb{R}^p$.

THEOREM 3.1. *(Weak Duality.) Assume that for all feasible x for (MOP) and all feasible (y, u, v) for (DMOP), any of the following holds:*

(a) (f, vg) is strong pseudoquasi-type I at y with respect to η and $u > 0$;
(b) (f, vg) is weak strictly pseudoquasi-type I at y with respect to η;
(c) (f, vg) is weak strictly pseudo-type I at y with respect to η.

Then the following cannot hold:

$$f(x) \leq f(y). \tag{3.16}$$

Proof. Suppose contrary to the result of the theorem that

$$f(x) \leq f(y). \tag{3.17}$$

hold. Since (y, u, v) is feasible for (DMOP), it follows that

$$-vg(y) \leqq 0. \tag{3.18}$$

By hypothesis (a) i.e (f, vg) is strong pseudoquasi-type I, (3.17) and (3.18) imply

$$(\nabla f(y))\eta(x, y) \leq 0, \tag{3.19}$$

$$v\nabla g(y)\eta(x, y) \leqq 0. \tag{3.20}$$

Since $u > 0$, the above inequalities give

$$[u\nabla f(y) + v\nabla g(y)]\eta(x, y) < 0, \tag{3.21}$$

which contradicts (3.11).

By hypothesis (b) i.e (f, vg) is weak strictly pseudoquasi-type I, (3.17) and (3.18) imply

$$(\nabla f(y))\eta(x, y) < 0, \tag{3.22}$$

$$v\nabla g(y)\eta(x, y) \leqq 0. \tag{3.23}$$

Since $u \geq 0$, (3.22) and (3.23) imply (3.21), again contradicting (3.11).

By hypothesis (c) i.e (f, vg) is weak strictly pseudo-type I, (3.17) and (3.18) imply

$$(\nabla f(y))\eta(x, y) < 0, \tag{3.24}$$
$$v\nabla g(y)\eta(x, y) < 0. \tag{3.25}$$

Since $u \geqq 0$, (3.24) and (3.25) imply (3.21), again contradicting (3.11). □

COROLLARY 3.1. *Assume weak duality (Theorem 3.1) holds between (MOP) and (DMOP). If* $(\mathring{y}, \mathring{u}, \mathring{v})$ *is feasible for (DMOP) such that* $\mathring{y}$ *is feasible for (MOP), then* $\mathring{y}$ *is efficient solution for (MOP) and* $(\mathring{y}, \mathring{u}, \mathring{v})$ *is efficient solution for (DMOP).*

Proof. The proof of this corollary is the same as that of Corollary 2 of Egudo (1989). □

THEOREM 3.2. *(Strong Duality) Let* $\mathring{x}$ *be efficient solution for (MOP) and assume that* $\mathring{x}$ *satisfies a constraint qualification (Marusciac (1982)) for (MOP). Then there exist* $\mathring{u} \in \mathbb{R}^p$ *and* $\mathring{v} \in \mathbb{R}^m$ *such that* $(\mathring{x}, \mathring{u}, \mathring{v})$ *is feasible for (DMOP). If also weak duality (Theorem 3.1) holds between (MOP) and (DMOP) then* $(\mathring{x}, \mathring{u}, \mathring{v})$ *is efficient solution for (DMOP).*

Proof. Since $\mathring{x}$ is efficient solution for (MOP) and satisfies a constraint qualification (Marusciac (1982)) for (MOP), then from Kuhn-Tucker necessary conditions (Marusciac (1982)) we obtain $\mathring{u} > 0$ and $\mathring{v} \geqq 0$ such that

$$\mathring{u}\nabla f(\mathring{x}) + \mathring{v}\nabla g(\mathring{x}) = 0,$$
$$vg(\mathring{x}) = 0$$

The vector $\mathring{u}$ may be normalized according to $\mathring{u}e = 1$, $\mathring{u} > 0$, which gives that the triplet $(\mathring{x}, \mathring{u}, \mathring{v})$ is feasible for (DMOP). The efficiency of $(\mathring{x}, \mathring{u}, \mathring{v})$ for (DMOP) now follows from Corollary 3.1. □

Now we state and prove our converse duality theorem of Mond–Weir vector type duality.

THEOREM 3.3. *(Converse Duality) Let* $(\mathring{x}, \mathring{u}, \mathring{v})$ *be efficient solution for (DMOP), and let the hypotheses of Theorem 3.1 hold. If the* $n \times n$ *Hessian matrix* $\nabla^2[\mathring{u} f(\mathring{x}) + \mathring{v}g(\mathring{x})]$ *is negative-definite and if* $\nabla \mathring{v}g(\mathring{x}) \neq 0$ *, then* $\mathring{x}$ *is efficient solution for (MOP).*

Proof. Since $(\mathring{x}, \mathring{u}, \mathring{v})$ is efficient solution for (DMOP) then the following Fitz John conditions hold (Da Cunha and Polak (1987)): there exists $\tau \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^n$,

$\alpha \in I\!R$, $\nu \in \mathbb{R}^m$, and $\mu \in \mathbb{R}^p$, such that

$$-(\nabla f(\mathring{x}))^t\tau + \nabla^2[\mathring{u} f(\mathring{x}) + \mathring{v} g(\mathring{x})]\lambda$$
$$-\alpha\nabla\mathring{v} g(\mathring{x}) = 0, \tag{3.26}$$
$$\nabla f(\mathring{x})\lambda - \mu = 0, \tag{3.27}$$
$$\nabla g(\mathring{x})\lambda - \alpha g(\mathring{x}) - \nu = 0, \tag{3.28}$$
$$\alpha\mathring{v} g(\mathring{x}) = 0, \tag{3.29}$$
$$\nu^t\mathring{v} = 0, \tag{3.30}$$
$$\mu^t\mathring{u} = 0, \tag{3.31}$$
$$\mathring{u}\nabla f(\mathring{x}) + \mathring{v}\nabla g(\mathring{x}) = 0, \tag{3.32}$$
$$\mathring{v} g(\mathring{x}) \geqq 0, \tag{3.33}$$
$$\mathring{v} \geqq 0, \tag{3.34}$$
$$\mathring{u} \geqq 0, \tag{3.35}$$
$$\mathring{u} e = 1, \tag{3.36}$$
$$(\tau, \alpha, \nu, \mu) \geqq 0. \tag{3.37}$$
$$(\tau, \lambda, \alpha, \nu, \mu) \neq 0. \tag{3.38}$$

Multiplying (3.27) by $\mathring{u}$ and using (3.31), multiplying (3.28) by $\mathring{v}$ and using (3.30) and (3.29), we obtain

$$\mathring{u}(\nabla f(\mathring{x}))^t\lambda = 0, \tag{3.39}$$
$$\mathring{v}(\nabla g(\mathring{x}))^t\lambda = 0. \tag{3.40}$$

Premultiplying (3.26) by λ^t and using (3.40), we have

$$-\lambda^t(\nabla f(\mathring{x}))^t\tau + \lambda^t\nabla^2[\mathring{u} f(\mathring{x}) + \mathring{v} g(\mathring{x})] = 0, \tag{3.41}$$

or

$$-\mu^t\tau + \lambda^t\nabla^2[\mathring{u} f(\mathring{x}) + \mathring{v} g(\mathring{x})]\lambda = 0. \tag{3.42}$$

We now claim that

$$\tau \neq 0. \tag{3.43}$$

Otherwise, from (3.42), we have

$$\lambda^t\nabla^2[\mathring{u} f(\mathring{x}) + \mathring{v} g(\mathring{x})]\lambda = 0. \tag{3.44}$$

Since $\nabla^2[\mathring{u} f(\mathring{x}) + \mathring{v} g(\mathring{x})]$ is assumed negative-definite, $\lambda = 0$. Therefore, from (3.26) we have

$$-\alpha\nabla\mathring{v} g(\mathring{x}) = 0. \tag{3.45}$$

Using the fact that $\nabla \overset{\circ}{v} g(\overset{\circ}{x}) \neq 0$, from (3.45) we obtain $\alpha = 0$, from (3.27), $\mu = 0$ and from (3.28), $\nu = 0$ contradicting (3.38). Hence, (3.43) holds.

From (3.42) and using (3.37) we obtain

$$\lambda^t \nabla^2 [\overset{\circ}{u} f(\overset{\circ}{x}) + \overset{\circ}{v} g(\overset{\circ}{x})] \lambda = \mu^t \tau \geqq 0. \tag{3.46}$$

Since $\nabla^2 [\overset{\circ}{u} f(\overset{\circ}{x}) + \overset{\circ}{v} g(\overset{\circ}{x})]$ is assumed negative-definite, $\mu^t \tau = 0$. Hence $\lambda = 0$; therefore, from (3.26) we have

$$(\nabla f(\overset{\circ}{x}))^t \tau = -\alpha (\nabla g(\overset{\circ}{x}))^t \overset{\circ}{v}, \tag{3.47}$$

From (3.32), we have

$$-(\nabla f(\overset{\circ}{x}))^t \overset{\circ}{u} = (\nabla g(\overset{\circ}{x}))^t \overset{\circ}{v}. \tag{3.48}$$

From (3.47) and (3.48), we get

$$(\tau^t - \alpha u^{0^t})(\nabla g(\overset{\circ}{x}))^t \overset{\circ}{v} = 0. \tag{3.49}$$

Using the hypothesis that, the vector $\nabla \overset{\circ}{v} g(\overset{\circ}{x}) \neq 0$, from (3.49) we get

$$\tau = \alpha \overset{\circ}{u}. \tag{3.50}$$

Hence $\alpha \neq 0$, because $\tau \neq 0$. Using (3.50) with $\lambda = 0$ in (3.28), we obtain

$$g(\overset{\circ}{x}) \leqq 0, \tag{3.51}$$

which shows that $\overset{\circ}{x}$ is feasible for the primal. Therefore, using Corollary 3.1 and the hypothesis of the theorem, we have that $\overset{\circ}{x}$ is efficient solution for (MOP). □

4. Generalized Mond–Weir Duality

We shall continue our discussion of duality for (MOP) in the present section by indroducing a general dual problem for (MOP) and proving weak and strong duality theorems under the weaker invexity assumptions.

We consider the following general Mond–Weir (1981) type dual problem

$$
\begin{aligned}
(GMOP) \qquad & \text{maximize} \quad f(y) + v_{J_0} g_{J_0}(y) e, & \\
& \text{subject to} \quad u \nabla f(y) + v \nabla g(y) = 0, & (4.52)\\
& v_{J_t} g_{J_t}(y) \geqq 0, \quad 1 \leqq t \leqq r, & (4.53)\\
& v \geqq 0, & (4.54)\\
& u \geqq 0, & (4.55)\\
& u^t e = 1; & (4.56)
\end{aligned}
$$

where $e = (1, 1, ..., 1)^t \in \mathbb{R}^p$ and J_t, $0 \leqq t \leqq r$ are partitions of the set P.

THEOREM 4.1. (Weak Duality). *Assume that for all feasible x for (MOP) and all feasible (y, u, v) for (GMOP),*

(a) $u > 0$, *and* $(f + v_{J_0} g_{J_0}(.)e, v_{J_t} g_{J_t}(.))$ *is strong pseudo-type I at y with respect to η for any t, $1 \leqq t \leqq r$;*

(b) $(f + v_{J_0} g_{J_0}(.)e, v_{J_t} g_{J_t}(.))$ *is weak strictly pseudoquasi-type I at y with respect to η for any t, $1 \leqq t \leqq r$;*

(c) $(f + v_{J_0} g_{J_0}(.)e, v^t_{J_t} g_{J_t}(.))$ *is weak strictly pseudo-type I at y with respect to η for any t, $1 \leqq t \leqq r$;*

then the following cannot hold:

$$f(x) \leq f(y) + v_{J_0} g_{J_0}(y)e. \tag{4.57}$$

Proof. Suppose to the contrary that (4.57) hold. Since x is feasible for (MOP) and $v \geqq 0$, (4.57) imply

$$f(x) + v_{J_0} g_{J_0}(x)e \leq f(y) + v_{J_0} g_{J_0}(y)e. \tag{4.58}$$

Also, from (4.53) we have

$$-v_{J_t} g_{J_t}(y) \leqq 0, \qquad \text{for all } 1 \leqq t \leqq r. \tag{4.59}$$

Using hypothesis **(a)**, we see that (4.58) and (4.59) together give

$$\begin{aligned}&(\nabla f(y) + v_{J_0} \nabla g_{J_0}(y)e)\eta(x, y) \leq 0,\\ &(v_{J_t} \nabla g_{J_t}(y))\eta(x, y) \leqq 0, \qquad \forall\, 1 \leqq t \leqq r.\end{aligned}$$

Since $u > 0$, the above inequalities give

$$[u\nabla f(y) + \sum_{t=0}^{r} v_{J_t} \nabla g_{J_t}(y)]\eta(x, y) < 0. \tag{4.60}$$

Since $J_0, J_1 \ldots, J_r$ are partitions of P, (4.60) is equivalent to

$$[u\nabla f(y) + v\nabla g(y)]\eta(x, y) < 0, \tag{4.61}$$

which contradicts (4.52).

Using hypothesis **(b)**, we see that (4.58) and (4.59) together give

$$\begin{aligned}&(\nabla f(y) + v_{J_0} \nabla g_{J_0}(y)e)\eta(x, y) < 0,\\ &(v_{J_t} \nabla g_{J_t}(y))\eta(x, y) \leqq 0, \qquad \forall\, 1 \leqq t \leqq r.\end{aligned}$$

Since $u \geq 0$, the above inequalities give

$$[u\nabla f(y) + \sum_{t=0}^{r} v_{J_t} \nabla g_{J_t}(y)]\eta(x, y) < 0, \tag{4.62}$$

and then again we have (4.61). Also we obtain a contradiction.

Suppose now that **(c)** is satisfied. Again from (4.58) and (4.59) it follows that

$$(\nabla f(y) + v_{J_0} \nabla g_{J_0}(y)e)\eta(x, y) < 0,$$
$$(v_{J_t} \nabla g_{J_t}(y))\eta(x, y) < 0, \qquad \forall\, 1 \leqq t \leqq r.$$

Since $u \geq 0$, the above inequalities imply (4.61), again contradicting (4.52). □

COROLLARY 4.1. *Assume weak duality (Theorem 4.1) holds between (MOP) and (GMOP). If $(\mathring{y}, \mathring{u}, \mathring{v})$ is feasible for (GMOP) with $v_{J_0} g_{J_0}(\mathring{y}) = 0$ and $\mathring{y}$ is efficient solution for (MOP) and $(\mathring{y}, \mathring{u}, \mathring{v})$ is efficient solution for (GMOP).*

Proof. The proof of this corollary is the same as that of Corollary 1 of Egudo (1989). □

THEOREM 4.2. (Strong Duality) *Let $\mathring{x}$ be an efficient solution for (MOP) and assume that $\mathring{x}$ satisfies a generalized constraint qualification (Maeda (1994)); then there exist $\mathring{u} \in \mathbb{R}^m$ and $\mathring{v} \in \mathbb{R}^p$ such that $(\mathring{y}, \mathring{u}, \mathring{v})$ is feasible for (GMOP) and $v_{J_0} g_{J_0}(\mathring{x}) = 0$. If also weak duality (Theorem 4.1) holds between (MOP) and (GMOP) then $(\mathring{y}, \mathring{u}, \mathring{v})$ is efficient solution for (MOP).*

Proof. Since $\mathring{x}$ is efficient solution for (MOP) and satisfies a generalized constraint qualification (Maeda (1994)), by Kuhn–Tucker necessary conditions (Maeda (1994)) there exists $\mathring{u} > 0$ and $\mathring{v} \geqq 0$ such that

$$\mathring{u} \nabla f(\mathring{x}) + \mathring{v} \nabla g(\mathring{x}) = 0, \tag{4.63}$$
$$\mathring{v}_i g_i(\mathring{x}) = 0, \qquad \forall\, 1 \leqq i \leqq p. \tag{4.64}$$

The vector $\mathring{u}$ may be normalized according to $\mathring{u}e = 1$, $\mathring{u} > 0$, which gives that the triplet $(\mathring{x}, \mathring{u}, \mathring{v})$ is feasible for (GMOP). From (4.64) we obtain $v_{J_0} g_{J_0}(\mathring{x}) = 0$. Efficiency of $(\mathring{x}, \mathring{u}, \mathring{v})$ for (GMOP) follows from Corollary 4.1. □

5. Conclusion

In this paper, we have defined new classes of functions called weak strictly pseudoquasi-type I and strong pseudoquasi-type I by relaxing the definitions of type I, weak strictly pseudoconvex (Marusciac (1982)), and strong pseudoconvex (Aghezzaf and Hachimi (1998)) functions. Similarly, the classes of weak quasi-strictly-pseudo-type I and weak strictly pseudo-type I functions are introduced as a generalization of quasipseudo-type I and strictly pseudo-type I functions (Kaul et al. (1994)). We have obtained various duality results for a nonlinear multiobjective programming problem involving the above classes of functions. Another result, is the relaxation of the assumption of the scalarization of the objective functions made in (Kaul et al. (1994)), Theorems 4.6–4.8) and (Giorgi and Guerraggio (1998)), Theorem 7) which was not required for our results.

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